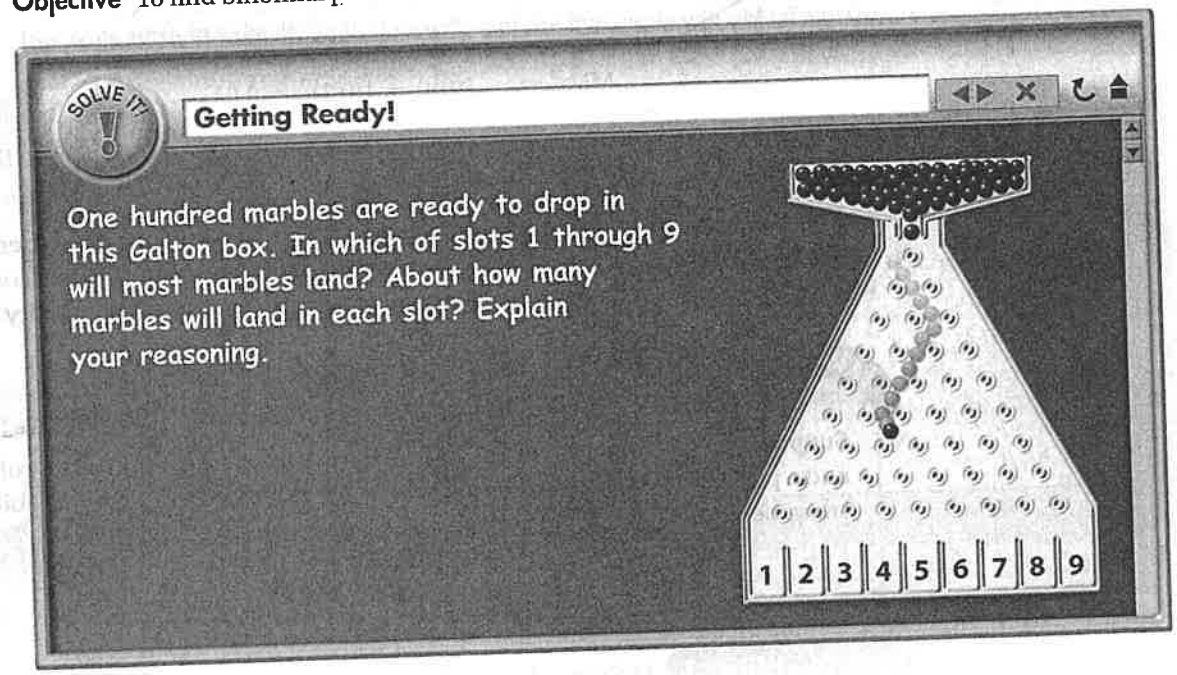
11-9 Binomial Distributions

Content Standard

Extends S.CP.9 Use permutations and combinations to compute probabilities of compound events and solve problems.

Objective To find binomial probabilities and to use binomial distributions



Dynamic Activity
Binomial Probability

Lesson Vocabulary
- binomial experiment
- binomial probability
- Binomial Theorem

At each level of a Galton box, a marble can take one of two possible paths.

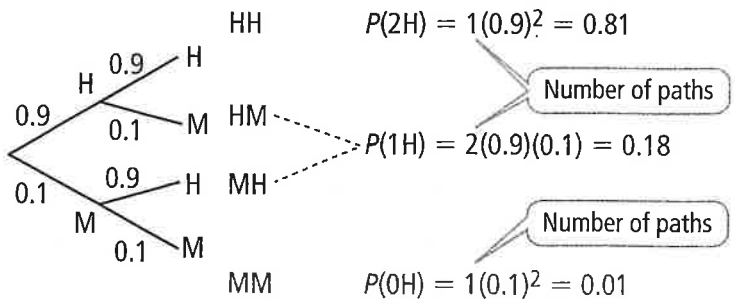
Essential Understanding You can use binomial probabilities in situations involving two possible outcomes.

take note **Key Concept Binomial Experiment**

A **binomial experiment** has these important features:

- There are a fixed number of trials.
- Each trial has two possible outcomes.
- The trials are independent.
- The probability of each outcome is constant throughout the trials.

Recall from Lesson 11-4, that you can use a tree diagram to find probabilities. The tree diagram on the following page shows different outcomes and probabilities for a basketball player shooting two free throws. It is known that this player is a good shooter, having hit (H) about 90% of the free throws so far this season.



Suppose the player needs to make one of two free throws to win a game. The purple and red labels show $P(2 \text{ hits}) + P(1 \text{ hit}) = 0.81 + 0.18 = 0.99$.

The basketball player shoots 2 free throws—each independent of the other (pressure notwithstanding). The player will succeed on 0, 1, or 2 of them. You can compute the probabilities shown in the tree diagram by using the **binomial probability** formula.

take note

Key Concept Binomial Probability

Suppose you have n repeated independent trials, each with a probability of success p and a probability of failure q (with $p + q = 1$). Then the **binomial probability** of x successes in the n trials can be found by the following formula.

$$P(x) = {}_nC_x p^x q^{n-x}$$

Problem 1 Using a Formula to Find Probabilities

Merchandising **As part of a promotion, a store is giving away scratch-off cards. Each card has a 40% chance of awarding a prize. Suppose you have five cards. Find the probability that exactly four of the five cards will reveal a prize.**

Know
- The number of trials n
- The number of successes x
- The probability of success p

Need
- The probability of failure q
- The probability of picking exactly 4 winning cards

Plan
- Decide that this is binomial probability.
- Find the probability of failure q.
- Use the formula for binomial probability.

Determine if this a binomial experiment.

- The situation involves 5 repeated trials—5 cards selected at random.
- Each trial has two possible outcomes: It is a winner or it is not.
- The probability of success is constant, 0.4, throughout the trials.
- The trials are independent. The outcome of scratching one card does not affect the probability of any of the other cards revealing a prize.

Think

How can you find ${}_nC_x$ using your calculator?

${}_nC_x = \frac{n!}{x!(n-x)!}$

On a graphing calculator, use MATH and ${}_nC_r$ in the PRB menu.

This is a binomial experiment with $n = 5, x = 4, p = 0.4$, and $q = 1 - p = 0.6$.

$P(x) = {}_nC_x p^x q^{n-x}$

$P(4) = {}_5C_4(0.4)^4(0.6)^1$ Substitute.

$= 5(0.4)^4(0.6)^1$ Evaluate ${}_5C_4$.

≈ 0.08 Simplify.

The probability is about 0.08 that exactly 4 of the five cards will reveal a prize.

Got It? **1.** What is the probability that the number of cards that reveal a prize is 0? 1? 2? 3? 5?

The Binomial Theorem (Lesson 5-7) says that for every positive integer n,

$$(a + b)^n = P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$$

where $P_0, P_1, \ldots, P_n$ are the numbers in the nth row of Pascal's Triangle.

For that row, it is possible to show that $P_i = {}_nC_i$. Thus, you can state the **Binomial Theorem** using combinations.

Take Note

Key Concept Binomial Theorem

For every positive integer n,

$$(a + b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_{n-1}ab^{n-1} + {}_nC_nb^n$$

Problem 2 Expanding Binomials

Use the Binomial Theorem to solve.

A What is the binomial expansion of $(x + y)^5$?

Use the Binomial Theorem with $a = x, b = y$, and $n = 5$.

$(x + y)^5 = {}_5C_0x^5 + {}_5C_1x^4y + {}_5C_2x^3y^2 + {}_5C_3x^2y^3 + {}_5C_4xy^4 + {}_5C_5y^5$

$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ Substitute for the ${}_nC_i$.

Think

Which ${}_4C_i$ do you use in the third term?

You use ${}_4C_0$, not ${}_4C_1$, for the first term. Therefore, use ${}_4C_2$ in the third term.

B What is the third term of $(2x - 3y)^4$?

The third term of the binomial expansion is ${}_4C_2a^{4-2}b^2$.

${}_4C_2a^{4-2}b^2 = {}_4C_2(2x)^2(-3y)^2$ Substitute $a = 2x$ and $b = -3y$.

$= 6(4x^2)(9y^2)$ Evaluate ${}_4C_2$.

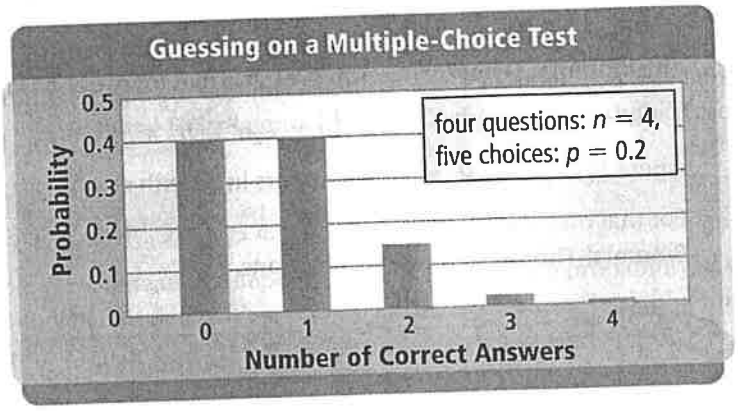
$= 216x^2y^2$ Simplify.

Got It? **2.** What is the binomial expansion of $(3x + y)^4$?

Now you can apply the Binomial Theorem to binomial probabilities. To find the full probability distribution for a binomial experiment, expand the binomial $(p + q)^n$. For example, suppose you guess on four questions of a five-choice multiple-choice test. For four questions, $n = 4$, $P(\text{guessing correctly}) = \frac{1}{5}$, so $p = 0.2$, and $q = 0.8$.

	4 correct		3 correct		2 correct		1 correct		0 correct
$(p + q)^4 =$	$1p^4$	$+$	$4p^3q$	$+$	$6p^2q^2$	$+$	$4pq^3$	$+$	$1q^4$
$=$	$(0.2)^4$	$+$	$4(0.2)^3(0.8)$	$+$	$6(0.2)^2(0.8)^2$	$+$	$4(0.2)(0.8)^3$	$+$	$(0.8)^4$
$=$	0.0016	$+$	0.0256	$+$	0.1536	$+$	0.4096	$+$	0.4096

You can display the distribution of binomial probabilities as a graph.



Problem 3 Applying Binomial Probability

Manufacturing **Each hour at a cell phone factory, Quality Control (QC) tests the durability of four randomly selected phones. If more than one fails, QC rejects the entire production for that hour. If in one hour, 95% of the phones made are acceptable, what is the probability that QC rejects that hour's phone production?**

Think

What is a "success" in one trial of this binomial experiment?
Success in this experiment means that a phone fails the test.

Write the binomial expansion of $(p + q)^n$ with $n = 4$, $p = 0.05$, and $q = 0.95$.

	4 fail		3 fail		2 fail		1 fail		0 fail
$(p + q)^4 =$	p^4q^0	$+$	$4p^3q^1$	$+$	$6p^2q^2$	$+$	$4p^1q^3$	$+$	p^0q^4

$$= (0.05)^4 + 4(0.05)^3(0.95)^1 + 6(0.05)^2(0.95)^2 + 4(0.05)^1(0.95)^3 + (0.95)^4$$

$$\approx 0.000006 + 0.000475 + 0.013538 + 0.171475 + 0.814506$$

$$\text{Probability (4, 3, or 2 phones fail)} \approx 0.000006 + 0.000475 + 0.013538$$

$$\approx 0.014019, \text{ or about } 1.4\%$$

There is about a 1.4% chance that QC will reject the phones produced in the last hour.

Got It? **3.** A multiple-choice quiz has five questions. Each question has four answer choices. If you guess every answer, what is the probability of getting at least three correct?

Lesson Check

Do you know HOW?

Find the probability of x successes in n trials for the given probability of success p on each trial.

1. $x = 2, n = 6, p = 0.4$
2. $x = 6, n = 9, p = 0.5$

Find the indicated term of each binomial expansion.

3. fourth term of $(c + d)^6$
4. second term of $(x - 2y)^5$
5. What is the probability of 2 successes in 4 trials of an experiment if the probability of success of one trial is 0.3?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

6. **Vocabulary** Explain how flipping a coin 10 times meets all of the conditions for a binomial experiment.
7. **Error Analysis** A student finds the fifth term of the binomial expansion $(j - k)^7$. Describe and correct the error the student made.

$${}_nC_5a^{(n-5)}b^5 = {}_7C_5j^2(-k)^5$$
$$= -21j^2k^5$$

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

See Problem 1.

A Practice

Find the probability of x successes in n trials for the given probability of success p on each trial.

8. $x = 3, n = 8, p = 0.3$
9. $x = 4, n = 8, p = 0.3$
10. $x = 5, n = 10, p = 0.5$
11. $x = 5, n = 10, p = 0.1$
12. **Battery Life** A calculator contains four batteries. With normal use, each battery has a 90% chance of lasting for one year. What is the probability that all four batteries will last a year?

See Problem 2.

Expand each binomial.

13. $(a + b)^4$
14. $(m + 5n)^3$
15. $(3x + 2y)^5$
16. $(4c - d)^4$

Find the indicated term of each binomial expansion.

17. second term of $(2g + 2h)^7$
18. fifth term of $(x - y)^5$
19. first term of $(e + 3f)^6$
20. eighth term of $(3x - y)^8$

See Problem 3.

Use the binomial expansion of $(p + q)^n$ to calculate each binomial distribution.

21. $n = 6, p = 0.3$
22. $n = 6, p = 0.5$
23. $n = 6, p = 0.9$
24. $n = 8, p = 0.45$

Apply **25. Think About a Plan** One survey found that 80% of respondents eat corn on the cob in circles rather than from side to side. Assume that this sample accurately represents the population. What is the probability that, out of five people you know, at least two of them eat corn on the cob in circles?

- How can you find the probability that one person eats corn on the cob in circles?
- How does a probability distribution help you solve the problem?

STEM **26. Weather** A scientist hopes to launch a weather balloon on one of the next three mornings. For each morning, there is a 40% chance of suitable weather. What is the probability that there will be at least one morning with suitable weather?

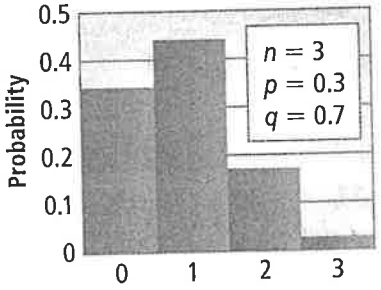
Marketing **A fruit company guarantees that 90% of the pineapples it ships will ripen within four days of delivery. Find each probability for a case containing 12 pineapples.**

27. All 12 are ripe within four days.

28. At least 10 are ripe within four days.

29. No more than 9 are ripe within four days.

30. Open-Ended Describe a situation that the graph might represent.



Sociology **A study shows that 50% of people in a community watch television during dinner. Suppose you select 10 people at random from this population. Find each probability.**

31. P(exactly 5 of the 10 people watch television during dinner)

32. P(exactly 6 of the 10 people watch television during dinner)

33. P(at least 5 of the 10 people watch television during dinner)

34. Writing Explain how a binomial experiment is related to a binomial expansion.

STEM **35. Quality Control** A company claims that 99% of its cereal boxes have at least as much cereal by weight as the amount stated on the box.

a. At a quality control checkpoint, one box out of a random sample of ten boxes falls short of its stated weight. What is the probability of this happening due to chance variation in box weights?

b. Reasoning Suppose three of ten boxes fail to have the claimed weight. What would you conclude? Explain.

36. Basketball Suppose you make 90% of your free throws and you attempt 3 free throws. Use the Binomial Theorem to calculate each probability.

a. You do not make any of them.

b. You only make 1 of them.

c. You only make 2 of them.

d. You make all of them.

STEM **37. Genetics** About 11% of the general population is left-handed. At a school with an average class size of 30, each classroom contains four left-handed desks. Does this seem adequate? Justify your answer.

38. Open-Ended Describe a binomial experiment that can be solved using the expression ${}_7C_2(0.6)^2(0.4)^5$.

39. Graph each probability distribution for $(p + q)^3$.

a. $p = 0.9, q = 0.1$

b. $p = 0.45, q = 0.55$

c. **Compare and Contrast** How are the graphs in parts (a) and (b) similar? How are they different?

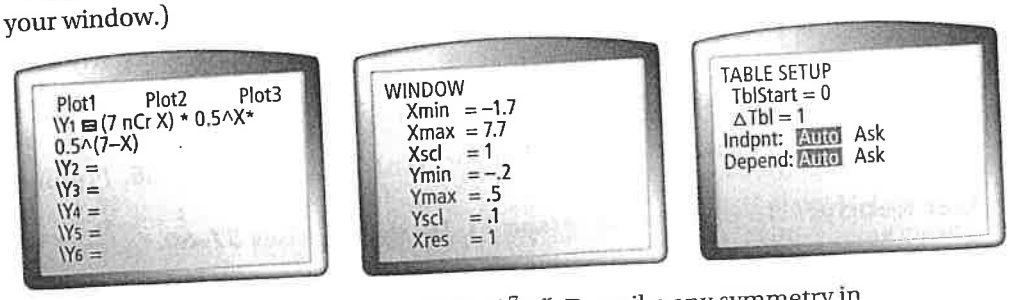
Challenge **Statistics A multiple-choice test has ten questions. Each question has five choices, with only one correct answer.**

40. Statisticians consider a "rare" event to have less than a 5% chance of occurring. According to this standard, what grades would be rare on this test if you guess? Justify your answer.

41. Design and conduct a simulation to model this situation. Gather results of simulations from your classmates. Do these results confirm the grades you identified as rare in Exercise 40? Explain.

42. Pascal's Triangle The nth row of Pascal's triangle has $n + 1$ terms. Find ${}_8C_4$. What row and term does this value represent in Pascal's Triangle? Use combinations to find the value of the 8th term of the 13th row of Pascal's triangle.

43. Graphing Calculator Enter the binomial probability formula as shown. Set the window and table shown. (To get integer values of x, you may need to adjust your window.)



a. Examine the graph of $y = {}_7C_x(0.5)^x(0.5)^{7-x}$. Describe any symmetry in the graph.

b. Verify the symmetry by displaying values of the function in table form.

c. Change the graph to $y = {}_7C_x(0.6)^x(0.4)^{7-x}$. Does this graph have any symmetry? Explain.

Standardized Test Prep

SAT/ACT

44. A survey shows that 60% of adults floss their teeth every day. In a random sample of ten adults, what is the probability that exactly six adults floss every day?

(A) 11% (B) 25% (C) 60% (D) 100%

45. Which of the statements about the following equation is correct?

$$\frac{b^2 - 4b + 3}{b - 3} = b - 1$$

(F) The equation is always true.

(G) The equation is always true, except when $b = 3$.

(H) The equation is never true.

(I) The equation is true when $b = 3$.

46. Which is the inverse of $f(x) = (x - 3)^2$?

(A) $f^{-1}(x) = \frac{x^2}{(3x - 1)^2}$

(B) $f^{-1}(x) = \pm\sqrt{x} + 3$

(C) $f^{-1}(x) = \frac{1}{(3x - 1)^2}$

(D) $f^{-1}(x) = \pm\sqrt{x - 3}$

47. If $\log 4 \approx 0.60206$ and $\log 5 \approx 0.69897$, what is the approximate value of $\log 80$?

(F) 0.2534 (G) 0.2914 (H) 1.903 (I) 11.1835

Extended Response

48. In a geometric sequence, $a_1 = 3$ and $a_4 = 192$. Explain how to find a_2 and a_3.

Mixed Review

Identify any bias in each survey question. See Lesson 11-8.

49. Do you agree that replacing that dog park with a beautiful new library would be better for our town?

50. Do you agree with the amendments to Proposition 39?

Find the vertices, foci, and asymptotes of each hyperbola. See Lesson 10-5.

51. $\frac{y^2}{49} - \frac{x^2}{25} = 1$ **52.** $4y^2 - 9x^2 = 36$ **53.** $64y^2 - 36x^2 = 576$

A standard number cube is tossed. Find each probability. See Lesson 11-3.

54. P(2 or greater than 3) **55.** P(6 or even) **56.** P(prime or 1)

Get Ready! To prepare for Lesson 11-10, do Exercises 57–60.

Find the mean and standard deviation for each data set. See Lessons 11-6 and 11-7.

57. 16, 20, 28, 25, 26, 33, 27, 22, 29, 18

58. 81, 78, 79, 80, 76, 88, 83, 90, 87, 76

59. 8.5, 7.9, 8.2, 9.0, 8.3, 9.1, 9.2

60. 23.5, 22.4, 25.6, 26.8, 28.1, 22.3, 24.5