11-1 Permutations and Combinations

Content Standard

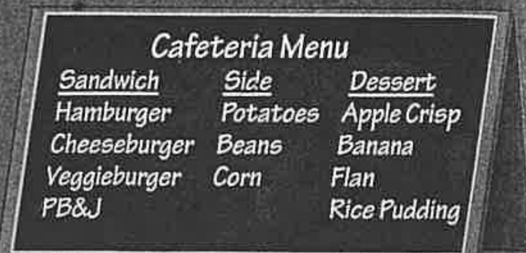
S.CP.9 Use permutations and combinations to compute probabilities of compound events and solve problems.

Objectives To count permutations
To count combinations

SOLVE IT!

Getting Ready!

For lunch, in how many different ways could you choose a sandwich, side dish, and dessert? Explain your reasoning.



MATHEMATICAL PRACTICES

It is fairly easy to count the ways you can pick items from a short list. But, sometimes you have so many choices that counting the possibilities is impractical.

Lesson Vocabulary
- Fundamental Counting Principle
- permutation
- *n* factorial
- combination

Essential Understanding You can use multiplication to quickly count the number of ways certain things can happen.

The **Fundamental Counting Principle** describes the method of using multiplication to count.

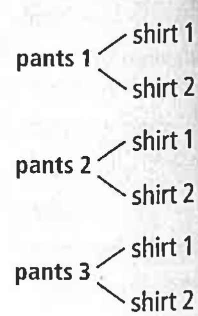
take note

Key Concept Fundamental Counting Principle

If event M can occur in m ways and is followed by event N that can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

Example 3 pants and 2 shirts give $3 \cdot 2 = 6$ possible outfits.

Here's Why It Works Making a tree diagram, you can see that there are 3 groups of 2 outfits, or $3 \cdot 2 = 6$ outfits, starting with the pants.



You can extend the Fundamental Counting Principle to three or more events.

Problem 1 Using the Fundamental Counting Principle

Motor Vehicles The photos show Maryland license plates in 2004 and 1912. How many more 2004-style license plates were possible than 1912-style plates?

Think
How many digits are in our number system? How many letters are in our alphabet?
There are 10 digits and 26 letters.

For the 2004 license plates, there were places for three letters and three digits. Number of possible 2004 license plates:

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$

For the 1912 license plates, there were places for four digits. Number of possible 1912 license plates:

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$

$17{,}576{,}000 - 10{,}000 = 17{,}566{,}000$ Find the difference.

There were 17,566,000 more 2004-style license plates possible than 1912-style plates.

Got It? 1. In 1966, one type of Maryland license plate had two letters followed by four digits. How many of this type of license plate were possible?

A **permutation** is an arrangement of items in a particular order. Suppose you wanted to find the number of ways to order three items. There are 3 ways to choose the first item, 2 ways to choose the second, and 1 way to choose the third. By the Fundamental Counting Principle, there are $3 \cdot 2 \cdot 1 = 6$ permutations.

Using *factorial* notation, you can write $3 \cdot 2 \cdot 1$ as 3!, read "three factorial." For any positive integer n, **n factorial** is $n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. Zero factorial is $0! = 1$.

Problem 2 Finding the Number of Permutations of *n* Items

In how many ways can you file 12 folders, one after another, in a drawer?

Plan
What strategy can you use to help you determine the answer?
Act it out and you will see how many options you have at each step.

Use the Fundamental Counting Principle to count the number of permutations of 12 items. There are 12 ways to select the first folder, 11 ways to select the next folder, and so on. The total number of permutations is

$12! = 12 \cdot 11 \cdot \ldots \cdot 2 \cdot 1 = 479{,}001{,}600.$

There are 479,001,600 ways to file 12 folders in a drawer.

Got It? 2. In how many ways can you arrange 8 shirts on hangers in a closet?

Sometimes you are interested in the number of permutations possible using all of the objects from a set, but just a few at a time. You can still use the Fundamental Counting Principle or factorial notation.

Key Concept Number of Permutations

The number of permutations of n items of a set arranged r items at a time is

$${}_nP_r = \frac{n!}{(n-r)!} \text{ for } 0 \le r \le n.$$

Example ${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$

Problem 3 Finding ${}_nP_r$

Track **Ten students are in a race. First, second, and third places will win medals. In how many ways can 10 runners finish first, second, and third (no ties allowed)?**

Plan

If you use the Fundamental Counting Principle, how many numbers will you need to multiply together? You will multiply 3 numbers together to represent the possibilities of finishing first, second, and third.

Method 1 Use the Fundamental Counting Principle.

$10 \cdot 9 \cdot 8 = 720$

Method 2 Use the permutation formula.

There are $n = 10$ runners to arrange taking $r = 3$ at a time.

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{Use the formula.}$$

$$= \frac{10!}{(10-3)!} \quad \text{Substitute 10 for } n \text{ and 3 for } r.$$

$$= \frac{10!}{7!} = 720 \quad \text{Simplify.}$$

There are 720 ways that 10 runners can finish in first, second, and third places.

Got It? 3. **a.** In how many ways can 15 runners finish first, second, and third?

b. Reasoning In Problem 3, is the number of ways for runners to finish first, second, and third the same as the number of ways to finish eighth, ninth, and tenth? Explain.

Suppose in Problem 3 that the first three runners advance to the championship race. In that case, the order in which the first three runners cross the finish line does not matter. A selection in which order does not matter is a **combination**.

As with permutations, you can use a formula to find the number of combinations of n items chosen r at a time.

Key Concept Number of Combinations

The number of combinations of n items of a set chosen r items at a time is

$${}_nC_r = \frac{n!}{r!(n-r)!} \text{ for } 0 \le r \le n.$$

Example ${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \cdot 2!} = \frac{120}{6 \cdot 2} = 10$

Problem 4 Finding $_nC_r$

What is $_{13}C_4$, the number of combinations of 13 items taken 4 at a time?

Think	Write
You need the formula for number of combinations. Substitute 13 for *n* and 4 for *r*.	$_nC_r = \frac{n!}{r!(n-r)!}$ $_{13}C_4 = \frac{13!}{4!(13-4)!}$ $= \frac{13!}{4! \cdot 9!}$
Write out the factorial in the numerator to make it easier to divide. Remove common factors.	$= \frac{13 \cdot \cancel{12} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot \cancel{9!}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{9!}} = 715$

Got It? 4. What is the value of each expression?

a. $_8C_3$ **b.** $_9C_2$ **c.** $_{15}C_5$

When determining whether to use a permutation or combination, you must decide whether order is important.

Problem 5 Identifying Whether Order Is Important

Plan
How will you solve?
If order is important, use the formula $_nP_r = \frac{n!}{(n-r)!}$ and if order is not important, use the formula $_nC_r = \frac{n!}{r!(n-r)!}$.

For each situation, determine whether you should use a permutation or combination. What is the answer to each question?

A **A chemistry teacher divides his class into eight groups. Each group submits one drawing of the molecular structure of water. He will select four of the drawings to display. In how many different ways can he select the drawings?**

There is no reason why order is important. Use a combination.

$$_nC_r = \frac{n!}{r!(n-r)!};\qquad _8C_4 = \frac{8!}{4!(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{4!}} = 70$$

There are 70 ways to select the drawings.

B **You will draw winners from a total of 25 tickets in a raffle. The first ticket wins \$100. The second ticket wins \$50. The third ticket wins \$10. In how many different ways can you draw the three winning tickets?**

Values of the tickets depend on the order in which you draw them. Order is important. Use a permutation.

$$_nP_r = \frac{n!}{(n-r)!};\qquad _{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = 25 \cdot 24 \cdot 23 = 13{,}800$$

There are 13,800 ways you can draw the winning tickets.

Got It? 5. In Problem 5A, how many ways are possible for the teacher to select and arrange the four drawings from left to right on the wall?

Lesson Check

Do you know HOW?

Evaluate each expression.

1. ${}_6P_3$ 2. ${}_9P_4$ 3. ${}_5C_2$ 4. ${}_7C_5$

5. **Sports** How many different nine-player batting orders can be chosen from a baseball team of 16?

Do you UNDERSTAND? MATHEMATICAL PRACTICES

6. **Vocabulary** Explain the difference between permutations and combinations.

7. **Reasoning** Use the definition of permutation to show why 0! should equal 1.

8. **Open-Ended** Describe a situation in which the number of outcomes is given by ${}_9P_2$.

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

A Practice

9. You have five shirts and four pairs of pants. How many different ways can you arrange your shirts and pants into outfits? See Problem 1.

10. To create an entry code for a push-button door lock, you need to first choose a letter and then, three single-digit numbers. How many different entry codes can you create?

11. The prom committee has four sites available for the banquet and three sites for the dance. How many arrangements are possible for the banquet and dance?

Evaluate each expression. See Problem 2.

12. 5! 13. 10! 14. 13! 15. 5!3!

16. $\frac{12!}{6!}$ 17. 5(4!) 18. $\frac{10!}{7!3!}$ 19. $\frac{15!}{10!5!}$

20. **Automobiles** You should rotate tires on a car at regular intervals.
    a. In how many ways can four tires be arranged on a car?
    b. If a spare tire is included, how many arrangements are possible?

Evaluate each expression. See Problem 3.

21. ${}_8P_1$ 22. ${}_8P_2$ 23. ${}_8P_3$ 24. ${}_8P_4$

25. ${}_3P_2$ 26. ${}_5P_4$ 27. ${}_9P_6$ 28. ${}_7P_4$

29. **Scheduling** Fifteen students ask to visit a college admissions counselor. Each scheduled visit includes one student. In how many ways can ten time slots be assigned?

Evaluate each expression. See Problem 4.

30. ${}_6C_2$ 31. ${}_8C_5$ 32. ${}_4C_4$ 33. ${}_4C_3$

34. ${}_7C_3$ 35. $3({}_5C_4)$ 36. ${}_6C_2 + {}_6C_3$ 37. $\frac{{}_7C_4}{{}_9C_4}$

38. **Awards** There are eight swimmers in a competition where the top three swimmers advance. In how many ways can three swimmers advance?

For each situation, determine whether to use a permutation or a combination. Then solve the problem. **See Problem 5.**

39. How many different teams of 11 players can be chosen from a soccer team of 16?

40. Suppose you find seven equally useful articles related to the topic of your research paper. In how many ways can you choose five articles to read?

41. A salad bar offers eight choices of toppings for a salad. In how many ways can you choose four toppings?

B Apply

Assume *a* and *b* are positive integers. Determine whether each statement is *true* or *false*. If it is true, explain why. If it is false, give a counterexample.

42. $a! + b! = b! + a!$ **43.** $a!(b!c!) = (a!b!)c!$ **44.** $(a + b)! = a! + b!$

45. $(ab)! = a!b!$ **46.** $(a!)! = (a!)^2$ **47.** $(a!)^b = a^{(b!)}$

48. Think About a Plan You and your friends are picking up videos at a video store. You have selected 7 videos but will only have time to watch 3 videos together. How many different ways can you select the 3 videos to watch?
- Does the order in which the videos are selected make a difference?
- What formula should you use?

49. Security A car door lock has a five-button keypad. Each button has two numerals. The entry code 21914 uses the same button sequence as the code 11023. How many different five-button patterns are possible? You can use a button more than once.

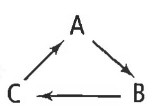
Ⓐ 120 Ⓑ 720 Ⓒ 3125 Ⓓ 5555

50. Consumer Issues A consumer magazine rates televisions by identifying two levels of price, five levels of repair frequency, three levels of features, and two levels of picture quality. How many different ratings are possible?

51. Writing In how many ways is it possible to arrange the two numbers *a* and *b* in an ordered pair? Explain why such a pair is called an *ordered* pair.

52. Reasoning Determine whether the statement ${}_nC_r = {}_nP_r$ is *always, sometimes,* or *never* true. Explain your reasoning.

53. There are 3!, or 6, arrangements of 3 objects. Consider the number of clockwise arrangements possible for objects placed in a loop, without a beginning or end.

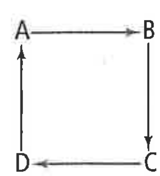


ABC, BCA, and CAB are all parts of one possible clockwise loop arrangement of the letters A, B, and C.

a. Find the number of clockwise loop arrangements possible for letters A, B, and C.

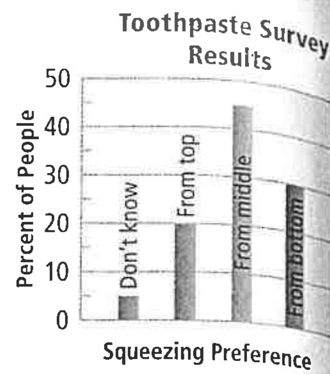
b. Use the diagram at the right to help find the number of loop arrangements possible for A, B, C, and D.

c. Write an expression for the number of clockwise loop arrangements for *n* objects.



Challenge

54. **Data Analysis** The bar graph at the right shows the results of 40 responses to a survey.
  a. Find the number of possible combinations of five people who squeeze toothpaste from the middle of the tube.
  b. Suppose five people are chosen at random from all the people who responded to the survey. How many combinations of five people are possible?



55. a. In how many ways can you choose three flags from a collection of seven different flags?
  b. In how many different orders can you arrange three flags?
  c. **Writing** You want to arrange three flags from a group of seven. Explain how you can use ${}_7C_3 \cdot 3!$ to create the permutation formula.

Standardized Test Prep

SAT/ACT

56. What is the value of ${}_7C_2$?

(A) 2520 (B) 49 (C) 42 (D) 21

57. What is the complete solution set of $\frac{3}{x^2 - 1} + \frac{4x}{x + 1} = \frac{1.5}{x - 1}$?

(F) 1, −1 (G) 1, 0.375 (H) 0.375 (I) 0.375, 3

58. Use a calculator to solve $-x^2 - 3x + 7 = 0$. Round to the nearest hundredth.

(A) −0.76, 4.76 (B) 0.76, 5.76 (C) −1.54, 4.54 (D) −4.54, 1.54

59. What is the center of the circle with equation $(x - 5)^2 + (y + 1)^2 = 81$?

(F) (5, 1) (G) (5, −1) (H) (−5, 1) (I) (−5, −1)

Short Response

60. What is the sum of the two infinite series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$ and $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}$?

Mixed Review

Identify the center, vertices, and foci for each ellipse. See Lesson 10-6.

61. $\frac{(x - 2)^2}{9} + \frac{(y - 1)^2}{25} = 1$

62. $\frac{(x - 1)^2}{49} + \frac{(y - 1)^2}{36} = 1$

Factor each expression completely. See Lesson 4-4.

63. $4x^2 - 8x + 4$

64. $-x^2 - 6x - 9$

65. $3x^2 - 75$

Get Ready! **To prepare for Lesson 11-2, do Exercises 66–68.**

Simplify each expression. See Lesson 11-1.

66. $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

67. $\frac{8 \cdot 7 \cdot 5 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$

68. $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$